Math Circles - Intro to Combinatorics - Winter 2024

Lecture 23

February 21th, 2024

1 Introduction

We began in the last lecture to look at patterns in the binomial coefficients. In the problem set you were asked to begin investigating how the coefficients changed in $(x+y)^n$ as we increase n as well as how the number of terms changes. Hopefully you observed that the number of terms of $(x+y)^{n+1}$ is one more than the number of terms of $(x+y)^n$. Using this we are going to arrange the coefficients of $(x+y)^n$ into a triangle.

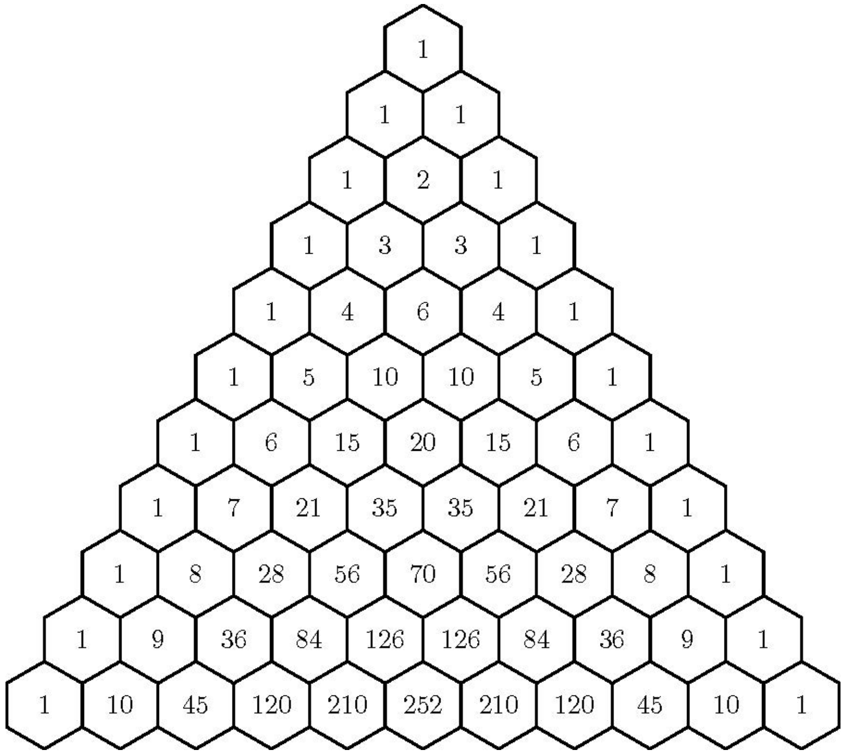
Taking your copy of the triangle diagram. Starting in the row with one hexagon write out the coefficient of $(x+y)^0$. Now moving on the the second row and $(x+y)^1$ write out the coefficients of the expansion in the hexagons starting with the highest power of x and following in decreasing order of powers of x. To clarify what this means we will do the third row, $(x+y)^2$ together. We know from last class that $(x+y)^2 = x^2+2xy+y^2$. So the coefficients in order of powers of x from highest to lowest are 1, 2, and 1. So we fill our three hexagons with these numbers left to right. Continue this until you have filled out the first six rows.

2 Pascal's Triangle

This triangle has a special name, Pascal's Triangle. It is named after French mathematician Blaise Pascal, though evidence shows it had been discovered prior to him centuries earlier in a number of countries including India, Italy, Persia, China, and Germany. In the 5th floor of MC there is a mathematics timeline and you can find a imagine of Pascal's triangle etched in stone on the timelines. The first known reference to this triangle was by a Persian mathematician Al-Karaji sometime between 953 and 1029 CE. It was discovered repeatedly by mathematicians and over the next few centuries and in 1665 it was published along with a number of it's properties by Pascal. In 1708 the triangle was referred to as Pascal's triangle and this became its standard western name. It is still known by other names around the world, including Yang Hui's triangle in China(who published details about the triangle but was not the first Chinese mathematician to discover the triangle) and Tartagalia's Triangle in Italy (after an Italian mathematician who published the first few rows of it in the 1500's). This is one of the fun things about math, especially in the past. Multiple people around the world can independently find and prove the same theorems. This has gotten less common with the internet as mathematicians can now research what other results have been published more easily. People still do independently discover results but it is not as common. Instead the greater access to what other mathematicians around the world are doing lets mathematicians collaborate more easily. So today we are going to investigate this triangle just like so many mathematicians of the past.

Now that we have filled out the first six rows we want to fill out the rest of the triangle. However, we want to do this without using the binomial theorem. Try to figure out how you can get from a row to the next row without using the binomial theorem. Try multiplying, adding, subtracting, or dividing terms to try and get between rows. If you find a pattern that works, fill out the rest of the rows on the sheet. Note while the triangle goes on infinitely we only have so much room on the page.

Now that we have spent some time looking at the pattern, you may have noticed that each new term can be obtained by adding together the two hexagons above it. We will fill out the whole triagnle together to check your answer and make sure we all agree on the triangle before moving on.



So what does this pattern tell us about combinations? Well each term can be represented by a combination. Two consecutive terms in a row come from the binomial theorem. We can write them as $\binom{n}{k}$ and $\binom{n}{k+1}$. So this pattern of addition in pascals triangle tell us that we should have $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$. Let's try and prove this.

Theorem 2.0.1 *For $n, k \in \mathbb{N}$, $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$*

Proof 1 *Let us write everything in terms of factorials.*

$$\begin{aligned}
\binom{n}{k}+\binom{n}{k+1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k+1)!(n-(k+1))!} \\
&= \frac{n!}{k!(n-k)(n-(k+1))!}+\frac{n!}{(k+1)k!(n-(k+1))!} \\
&= \frac{n!}{k!(n-k)(n-(k+1))!}\times\frac{k+1}{k+1}+\frac{n!}{(k+1)k!(n-(k+1))!}\times\frac{n-k}{n-k} \\
&= \frac{(k+1)n!+(n-k)n!}{(k+1)k!(n-k)(n-(k+1))!} \\
&= \frac{k\times n!+n!+n\times n!-k\times n!}{(k+1)!(n-k)!} \\
&= \frac{n!(n+1)}{(k+1)!((n+1)-(k+1))!} \\
&= \frac{(n+1)!}{(k+1)!((n+1)-(k+1)!}=\binom{n+1}{k+1}.
\end{aligned}$$

One important thing to note, we often refer the the first row of pascal's triangle as the 0th row. We start counting at zero, like in computer science, so that the number of our row corresponds to the degree on the binomial we are expanding.

3 Pattern's in Pascal's Triangle

Patterns and sequences of numbers are incredibly important in mathematics. Finding patterns is often how research begins. We love patterns so much that we even have a database to collect information about patterns in sequences of numbers called OEIS, the online encyclopedia of integer sequences.

You might recall that last week we discovered that the sum of the binomial coefficients for a given n is 2^n. Now we are going to see what other patterns we can find in the triangle. Follow along with the problem set to investigate different patterns in the triangle.

When looking for patters there are many different things you can try.

1. What patterns can you find in diagonals on the triangle?
2. Do you notice anything if you mark all the even number?
3. Or the odd numbers?
4. What about number that are divisible by three?
5. Do you notice anything special about the rows that correspond to prime numbers?
6. What happens if you alternate adding and subtracting the entries in a row of pascals triangle?

These are some ideas of patterns you can try to find in problem set 3. Work on the patterns suggested in problem set 3 and we will periodically stop to discuss the findings.